

Gaussian bounds and Collisions of variable speed random walks on lattices with power law conductances

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Abstract

We consider a weighted lattice \mathbb{Z}^d with conductance $\mu_e = |e|^{-\alpha}$. We show that the heat kernel of a variable speed random walk on it satisfies a two-sided Gaussian bound by using an intrinsic metric. We also show that when $d = 2$ and $\alpha \in (-1, 0)$, two independent random walks on such weighted lattice will collide infinite many times while they are transient.

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1 Introduction

In [16], Hebisch and Saloff-Coste proved that when a group has polynomial volume growth of order D , the heat kernel of a constant speed random walk on the group satisfies a two-sided Gaussian estimate, i.e.,

$$c_1 t^{-D/2} \exp\left(-c_2 \frac{\rho(x, y)^2}{t}\right) \leq p_t(x, y) \leq c_3 t^{-D/2} \exp\left(-c_4 \frac{\rho(x, y)^2}{t}\right).$$

where $\rho(x, y)$ is a metric on the group. Delmotte [12] gave equivalence of Gaussian bounds, parabolic Harnack inequalities, and the combination of volume regularity and Poincaré inequality. Later, there are many papers, such as [1, 2, 3, 4, 5, 25], showing that Gaussian bounds hold for lattice \mathbb{Z}^d with different random conductances. In this paper, we consider a deterministic weighted lattice which does not satisfy Poincaré inequalities for all (sufficiently large) balls or volume doubling property, show that a variable random walk on it also satisfies the two-side Gaussian bound, but with a metric which is not comparable to the Euclidean metric.

Let $\alpha \in \mathbb{R}$. For $x, y \in \mathbb{Z}^d$ with $|x - y|_1 = 1$, we set $\mu_{xy} = (|x|_\infty \vee |y|_\infty)^{-\alpha}$ for the conductance of (x, y) . For convenience, we set $\mu_{xy} = 0$ if x and y are not nearest neighbor.

Write $\mu_x = \sum_y \mu_{xy}$ and $\nu_x = (|x|_\infty \vee 1)^\alpha$ for each $x \in \mathbb{Z}^d$. Let $X = \{X_t : t \geq 0\}$ be a continuous time random walk on the lattice \mathbb{Z}^d with generator

$$\mathcal{L}f(x) = \frac{1}{\nu_x} \sum_{y \in \mathbb{Z}^d} (f(y) - f(x))\mu_{xy}.$$

Then X is a variable speed random walk waiting for an exponentially distributed time with mean $\frac{\nu_x}{\mu_x} \asymp |x|_\infty^{2\alpha}$ before jumping. The transition density of X with respect to ν is denoted by

$$p_t(x, y) = \frac{\mathbb{P}_x(X_t = y)}{\nu_y}.$$

To show the Gaussian bounds hold, we introduce a metric ρ of \mathbb{Z}^d . We call $x_0 \cdots x_m$ a path if $|x_{i+1} - x_i|_1 = 1$ for each $i < m$. Let $\rho(x, x) = 0$ for $x \in \mathbb{Z}^d$, and for $x, y \in \mathbb{Z}^d$ with $y \neq x$ set

$$\rho(x, y) = \min \left\{ \sum_{i=0}^m \nu_{z_i} : z_0 z_1 \cdots z_m \text{ is a path with } z_0 = x \text{ and } z_m = y \right\}.$$

Then there exists a constant $C = C(\alpha, d)$, such that

$$\frac{1}{\nu_x} \sum_{y \sim x} \rho(x, y)^2 \mu_{xy} \leq C \quad \text{for all } x. \quad (1.1)$$

Metrics satisfying (1.1) are called intrinsic metrics, see [14, 27]. One may expect that analogues of diffusion processes on manifolds hold using the intrinsic metrics for random walks on graphs. For $x \in \mathbb{Z}^d$ and $r \in \mathbb{R}^+$, write $B_\rho(x, r) = \{y \in \mathbb{Z}^d : \rho(x, y) \leq r\}$ for a ρ -ball. We extend ν to a measure on \mathbb{Z}^d and set

$$V_\rho(x, r) = \nu(B_\rho(x, r)).$$

Theorem 1.1 *Let $\alpha > -1$. Let $x, y \in \mathbb{Z}^d$ and $t > 0$. If $t < (\nu_x \vee \nu_y)\rho(x, y)$, then*

$$p_t(x, y) \leq c_1(\nu_x \nu_y)^{-1/2} \exp \left(-\frac{c_2 \rho(x, y)}{\nu_x \vee \nu_y} \left(1 \vee \log \left(\frac{(\nu_x \vee \nu_y)\rho(x, y)}{t} \right) \right) \right). \quad (1.2)$$

If $t \geq (\nu_x \vee \nu_y)\rho(x, y)$, then

$$p_t(x, y) \leq \frac{c_3}{\sqrt{V_\rho(x, t^{1/2})V_\rho(y, t^{1/2})}} \exp \left(-c_4 \frac{\rho(x, y)^2}{t} \right) \quad (1.3)$$

and

$$p_t(x, y) \geq \frac{c_5}{\sqrt{V_\rho(x, t^{1/2})V_\rho(y, t^{1/2})}} \exp \left(-c_6 \frac{\rho(x, y)^2}{t} \right). \quad (1.4)$$

Remark 1.2 (1) In Lemmas 2.2 and 2.4, we give the bounds of $\rho(x, y)$ and $V_\rho(x, t^{1/2})$, respectively.

(2) Note that if $\alpha < -1$ then $\sup_{x, y} \rho(x, y) < \infty$ and X will explode in a finite time. However, we still do not know whether the heat kernel of X has Gaussian bounds at the critical point $\alpha = -1$.

Next, we consider the collision problem of random walks on these weighted lattices. As usual, we say that two walks X and X' collide infinitely often if almost surely there exists a sequence of (random) times $\{t_i : i \geq 1\}$ with $\lim_i t_i = \infty$ such that $X_{t_i} = X'_{t_i}$ for all i . In [24], Pólya first studied whether two independent simple random walks on \mathbb{Z}^d collide infinitely often. He reduced it to the problem of a single walker returning to his starting point. Later Jain and Pruitt in [23] showed the Hausdorff dimension of the intersection of two independent stable processes, and Shieh in [26] gave a sufficient condition for infinitely collisions of Lévy processes in \mathbb{R} . However, if the walks are not on a homogeneous space, the problem will be complicated. Recently in [17], Hutchcroft and Peres use the Mass-Transport Principle to prove that a recurrent reversible random rooted graph has the infinite collision property. Examples that two recurrent random walks will never meet, were shown in [6, 7, 18]. Here, we give another example that two transient random walks will collision infinite often.

Theorem 1.3 *Let $\alpha > -1$. Let X' be an independent copy of X .*

- (1) *Process X is recurrent if and only if $\alpha \geq d - 2$.*
- (2) *If $d \leq 2$, then X and X' collide infinitely often.*
- (3) *If $d \geq 3$, then X and X' collide finitely often.*

Remark 1.4 *It is much interesting that X is not recurrent while X and X' collide infinitely often when $d = 2$ and $\alpha \in (-1, 0)$. Similarly, when $d \geq 3$ and $\alpha \geq d - 2$, X is recurrent while X and X' collide finitely often.*

In Section 2, we obtain some geometric properties of the weighted lattice \mathbb{Z}^d . In Section 3, we obtain an upper bound on $p_T(w, w)$ by using the approach of Barlow and Chen [4], which in turn is based on [19, 2]. In Section 4, we obtain the lower bounds of near diagonal transition probability by using the result of Delmette [12] directly and a chain argument. In Section 5, we give the proof of Theorem 1.1. Section 6 deals with the proof of Theorem 1.3 by the two-sided Gaussian bounds.

Throughout this paper, we use the notation c, c' etc to denote fixed positive constants which may vary on each appearance, and c_i to denote positive constants which are fixed in each argument. If we need to refer to constant c_1 of Lemma 2.1 elsewhere we will use the notation $c_{2.1.1}$. For any two functions f and g , we say $f \asymp g$ if there exists $c_i(\alpha, d) > 0$ such that $c_1 f \leq g \leq c_2 f$. For brevity, we write $|\cdot|_p$ for the L^p -norm of the Euclidean space \mathbb{R}^d , while $|\cdot|$ instead of $|\cdot|_\infty$ for the L^∞ -norm. Write $B(x, r) = \{y \in \mathbb{Z}^d : |y - x| \leq r\}$ for an L^∞ -ball.

2 Some geometric properties

Fix $\alpha > -1$ henceforth. In this section, we shall estimate the metric $\rho(x, y)$ and the volume $V_\rho(x, r)$, and give Poincaré inequalities. Let us begin with the volume of a path.

Lemma 2.1 *Let $z_0 \cdots z_n$ be a path with $\max\{|z_0|, |z_n|, |z_0 - z_n|_1\} \geq n \geq 1$. Then*

$$c_1 n (|z_0| \vee |z_n|)^\alpha \leq \sum_{i=0}^n \nu_{z_i} \leq c_2 n (|z_0| \vee |z_n|)^\alpha. \quad (2.1)$$

Proof. Without loss generality, we may assume that $|z_0| \geq |z_n|$ in the following. (Otherwise, relabel z_{n-k} with z_k for all k .) Then

$$|z_0 - z_n|_1 \leq d|z_0 - z_n| \leq d|z_0| + d|z_n| \leq 2d|z_0|.$$

Using the condition $\max\{|z_0|, |z_n|, |z_0 - z_n|_1\} \geq n \geq 1$, we get

$$|z_0| \geq \frac{n}{2d} \vee 1. \quad (2.2)$$

Since $z_0 \cdots z_n$ is a path, we have $|z_i - z_0| \leq i$ for each i . So, $\nu_{z_i} = (|z_i| \vee 1)^\alpha$ takes value between $(|z_0| + i)^\alpha$ and $((|z_0| - i) \vee 1)^\alpha$. Hence $\nu_{z_i} \geq c|z_0|^\alpha$ for $i \leq \frac{n}{4d}$, which implies

$$\sum_{i=0}^n \nu_{z_i} \geq \sum_{i \leq n/(4d)} \nu_{z_i} \geq c \lceil n/4d \rceil \nu_{z_0} \geq c'n|z_0|^\alpha = c'n(|z_0| \vee |z_n|)^\alpha. \quad (2.3)$$

We have proved the lower bound of (2.1). For the upper bound, we consider two cases.

Case I: $|z_0| \geq |z_n| \vee n$. Directly calculate

$$\begin{aligned} \sum_{i=0}^n \nu_{z_i} &\leq \sum_{i=0}^n ((|z_0| + i)^\alpha + ((|z_0| - i) \vee 1)^\alpha) \leq 2 \sum_{i=|z_0|-n}^{|z_0|+n} (i \vee 1)^\alpha \\ &\leq c_1 \int_{|z_0|-n}^{|z_0|+n} x^\alpha dx = \frac{c_1}{1+\alpha} ((|z_0| + n)^{\alpha+1} - (|z_0| - n)^{\alpha+1}). \end{aligned}$$

Since $\lim_{t \rightarrow 0+} ((1+t)^{\alpha+1} - (1-t)^{\alpha+1})t^{-1} = 2(\alpha+1)$, we obtain

$$\sup_{t \in (0,1]} |((1+t)^{\alpha+1} - (1-t)^{\alpha+1})t^{-1}| \leq c_2.$$

Substituting $t = \frac{n}{|z_0|} \leq 1$ into the above inequality gives

$$\sum_{i=0}^n \nu_{z_i} \leq \frac{c_1}{1+\alpha} ((|z_0| + n)^{\alpha+1} - (|z_0| - n)^{\alpha+1}) \leq \frac{c_1 c_2}{1+\alpha} n |z_0|^\alpha = cn(|z_0| \vee |z_n|)^\alpha.$$

Case II: $|z_n| \leq |z_0| < n$ and $|z_0 - z_n|_1 = n$. Then $z_0 \cdots z_n$ is an L^1 -geodesic, which implies $\{z_0, \dots, z_n\} \subset B(0, n)$ and $|\{i : z_i \in B(0, r)\}| \leq 2dr$ for each r . Write

$$k = \lceil \log_2 n \rceil, \quad T_0 = B(0, 1) \quad \text{and} \quad T_l = B(0, 2^l) - B(0, 2^{l-1}) \quad \text{for } l \geq 1.$$

Then

$$\begin{aligned} \sum_{i=0}^n \nu_{z_i} &= \sum_{l=0}^k \sum_{i: z_i \in T_l} \nu_{z_i} \leq c \sum_{l=0}^k 2^{l\alpha} |\{i : z_i \in T_l\}| \leq c \sum_{l=0}^k 2^{\alpha l} |\{i : z_i \in B(0, 2^l)\}| \\ &\leq c \sum_{l=0}^k 2^{\alpha l} (2d \cdot 2^l) = 2dc \sum_{l=0}^k 2^{(\alpha+1)l} \leq c' 2^{(1+\alpha)(k-1)} \leq c'n^{1+\alpha}. \end{aligned}$$

Since $\frac{n}{2d} \leq |z_0| < n$, we still have $\sum_{i=0}^n \nu_{z_i} \leq c_2 n(|z_0| \vee |z_n|)^\alpha$ and prove the lemma. \square

For $x \in \mathbb{Z}^d$ and $r \in \mathbb{R}^+$, we set

$$\rho_x(r) = (|x| \vee r)^\alpha r. \quad (2.4)$$

Then $\rho_x(\cdot)$ is strictly increasing and

$$\left(\frac{r}{s}\right)^{c_1} \leq \frac{\rho_x(r)}{\rho_x(s)} \leq \left(\frac{r}{s}\right)^{c_2}, \quad \forall \ r \geq s > 0. \quad (2.5)$$

A simple calculation gives, if $x, y \in \mathbb{Z}^d$ and $r \geq \kappa|x - y|$, then there exists $C = C(\alpha, \kappa) > 0$ such that

$$C^{-1}\rho_y(r) \leq \rho_x(r) \leq C\rho_y(r). \quad (2.6)$$

Set $\rho_x^{-1}(r) = (|x| \vee r^{1/(1+\alpha)})^{-\alpha}r$, which is the inverse function of ρ_x . Then $\rho_x^{-1}(\cdot)$ also satisfies (2.5) and (2.6).

Lemma 2.2 *Let $x, y \in \mathbb{Z}^d$. Let γ be an L^1 -geodesic path from x to y . Then*

$$\left\{ \rho(x, y), \sum_{u \in V(\gamma)} \nu_u, \sum_{(u, v) \in E(\gamma)} \mu_{uv}^{-1} \right\} \subset [c_1 \rho_x(|x - y|), c_1^{-1} \rho_x(|x - y|)].$$

Proof. By (2.6), we have $\rho_x(|x - y|) \asymp \rho_y(|x - y|)$. So, we may assume $|x| \geq |y|$ without loss generality. (Otherwise, exchange y with x .) Hence $|x| \geq \frac{1}{2}(|x| + |y|) \geq \frac{1}{2}|x - y|$, which implies

$$\rho_x(|x - y|) \asymp |x - y| \cdot |x|^\alpha. \quad (2.7)$$

Let $z_0 z_1 \cdots z_m$ be a ρ -geodesic path with $z_0 = x$ and $z_m = y$, then by Lemma 2.1,

$$\rho(x, y) \geq \frac{1}{2} \sum_{k=0}^{\lceil |x-y|/2 \rceil} \nu_{z_k} \geq c \lceil |x - y|/2 \rceil |x|^\alpha.$$

By the definition of $\rho(x, y)$, it is clear that $\sum_{u \in V(\gamma)} \nu_u \geq \rho(x, y)$. Moreover, by Lemma 2.1,

$$\sum_{u \in V(\gamma)} \nu_u \leq c|x - y|_1(|x| \vee |y|)^\alpha \leq 2dc|x - y||x|^\alpha.$$

Since $\mu_{uv}^{-1} \asymp (|u| \vee 1)^\alpha = \nu_u$ whenever $u \sim v$, we also have

$$\sum_{(u, v) \in E(\gamma)} \nu_{uv}^{-1} \asymp \sum_{u \in V(\gamma)} \nu_u$$

Combining these inequalities together, we complete the proof. \square

Since $\rho_x(r)$ is increasing in r , Lemma 2.2 immediately implies Corollary 2.3 as follows. Recall that $B_\rho(x, r)$ is a ρ -ball. One can compare it with an L^1 -ball.

Corollary 2.3 For any $x \in \mathbb{Z}^d$ and $r > 0$,

$$B(x, \rho_x^{-1}(c_1 r)) \subset B_\rho(x, r) \subset B(x, \rho_x^{-1}(c_2 r)).$$

Recall that $V_\rho(x, r)$ is the volume of $B_\rho(x, r)$. Set $V(x, r) = \nu(B(x, r))$, similarly.

Lemma 2.4 Let $x \in \mathbb{Z}^d$ and $r > 0$.

$$(1) \quad V(x, r) \asymp r^d(|x| \vee r)^\alpha \quad \text{if } r \geq 1.$$

$$(2) \quad V_\rho(x, r) \asymp V(x, \rho_x^{-1}(r)) \asymp \begin{cases} \nu_x & \text{if } r < \nu_x; \\ r^d |x|^{-(d-1)\alpha} & \text{if } \nu_x \leq r \leq |x|^{1+\alpha}; \\ r^{(d+\alpha)/(1+\alpha)} & \text{if } r > |x|^{1+\alpha}. \end{cases}$$

Proof. (1) Let x_1 be the first coordinate of x and set

$$\Lambda = \{s = (s_1, \dots, s_d) \in B(x, r) : s_1 = x_1\}.$$

Write $e_1 = (1, 0, 0, \dots, 0) \in \mathbb{Z}^d$. By Lemma 2.1, for each $s \in \Lambda$ we have

$$\sum_{l=-r}^r \nu_{s+le_1} \asymp r(|s - re_1| \vee |s + re_1|)^\alpha \asymp r(|s| \vee r)^\alpha \asymp r(|x| \vee r)^\alpha.$$

Hence,

$$V(x, r) = \sum_{s \in \Lambda} \sum_{l=-r}^r \nu_{s+le_1} \asymp |\Lambda| \cdot r(|x| \vee r)^\alpha \asymp r^d(|x| \vee r)^\alpha. \quad (2.8)$$

(2) Using (2.8) and Corollary 2.3, we get the desired result. \square

Lemma 2.5 Let $w \in \mathbb{Z}^d$ and $R \geq 1$. Then for any $x \in B(w, R)$ and $r \in [1, R]$,

$$V(w, R) \leq c_1 \left(\frac{R}{r}\right)^{c_1} V(x, r). \quad (2.9)$$

Especially, $V(w, R) \leq c_1 R^{c_1} \nu_x$.

Proof. It follows directly from Lemma 2.4 (1). \square

So, $\nu(B(w, R))$ satisfy the volume doubling property in any case. However, $\mu(B(w, R)) = \sum_{x \in B(w, R)} \mu_x$ do not satisfy the volume doubling property since $\mu(\mathbb{Z}^d) < \infty$ when $\alpha > d$.

In [28] Virág, extending the early result of [22], showed that Poincaré inequalities hold in any convex lattices. We shall apply their technique to our weighted lattices.

Lemma 2.6 *Let $x \in \mathbb{Z}^d$, $r > 0$. Then for any function f on $B(x, r)$,*

$$\min_a \sum_{u \in B(x, r)} (f(u) - a)^2 \nu_u \leq c_1 [\rho_x(r)]^2 \sum_{u, v \in B(x, r)} (f(u) - f(v))^2 \mu_{uv}. \quad (2.10)$$

Proof. If $r \in (0, 1)$ then $B(x, r) = \{x\}$ and (2.10) holds since both side of the inequality are zero. So, we may assume that $r \geq 1$ in the following.

By [28, Proposition 2], for each $u, v \in \mathbb{Z}^d$ we can choose a path γ_{uv} such that, (1) γ_{uv} is an L^1 -geodesic path from u to v ; (2) each site in γ_{uv} has L^∞ -distance less than 1 from the Euclidean line \overline{uv} . For $u, y \in \mathbb{Z}^d$, write

$$\Lambda_{uy} = \{s + z : s \in \gamma_{y, 2y-u}, |z| \leq 4, z \in \mathbb{Z}^d\}.$$

By the construction, we have

$$1_{\{y \in \gamma_{uv}\}} \leq 1_{\{v \in \Lambda_{uy}\}} + 1_{\{u \in \Lambda_{vy}\}} \quad \text{for all } u, v, y.$$

By Lemma 2.2,

$$\sum_{v \in \Lambda_{uy}} \nu_v \leq \sum_{s \in \gamma_{y, 2y-u}} \sum_{z \in \mathbb{Z}^d, |z| \leq 4} \nu_{s+z} \leq c \sum_{s \in \gamma_{y, 2y-u}} \nu_s \leq c' \rho_y(|y - u|).$$

So, if $u, y \in B(x, r)$, we can use (2.6) and get

$$\sum_{v \in \Lambda_{u, y}} \nu_v \leq c \rho_y(2r) \leq c' \rho_x(r).$$

By Lemma 2.2, if $u, v \in B(x, r)$ then

$$\sum_{(y, z) \in E(\gamma_{uv})} \mu_{yz}^{-1} \leq c \rho_u(|u - v|) \leq c \rho_x(r). \quad (2.11)$$

Therefore, writing $B = B(x, r)$,

$$\begin{aligned} \sum_{u \in B} (f(u) - \bar{f})^2 \nu_u &\leq \frac{1}{\nu(B)} \sum_{u, v \in B} (f(u) - f(v))^2 \nu_u \nu_v = \frac{1}{\nu(B)} \sum_{u, v \in B} \left(\sum_{(y, z) \in E(\gamma_{uv})} (f(y) - f(z)) \right)^2 \nu_u \nu_v \\ &\leq \frac{1}{\nu(B)} \sum_{u, v \in B} \left(\sum_{(y, z) \in E(\gamma_{uv})} (f(y) - f(z))^2 \mu_{yz} \right) \left(\sum_{(y, z) \in E(\gamma_{uv})} \mu_{yz}^{-1} \right) \nu_u \nu_v \\ &\leq \frac{c \rho_x(r)}{\nu(B)} \sum_{u, v \in B} \sum_{(y, z) \in E(\gamma_{uv})} (f(y) - f(z))^2 \mu_{yz} \nu_u \nu_v \\ &\leq \frac{c \rho_x(r)}{\nu(B)} \sum_{y, z \in B} (f(y) - f(z))^2 \mu_{yz} \sum_{u, v \in B} 1_{\{y \in \gamma_{u, v}\}} \nu_u \nu_v \end{aligned}$$

$$\begin{aligned}
&\leq \frac{c\rho_x(r)}{\nu(B)} \sum_{y,z \in B} (f(y) - f(z))^2 \mu_{yz} \left(\sum_{u \in B} \nu_u \sum_{v \in \Lambda_{u,y}} \nu_v + \sum_{v \in B} \nu_v \sum_{u \in \Lambda_{v,y}} \nu_u \right) \\
&\leq c' [\rho_x(r)]^2 \sum_{y,z \in B} (f(y) - f(z))^2 \mu_{yz},
\end{aligned}$$

where the second inequality is by the Cauchy-Schwarz inequality. \square

Lemma 2.7 *Let $w \in \mathbb{Z}^d$, $R \geq 1$ and $r \in (0, \rho_w(R)]$. Let $g : B(w, R) \rightarrow \mathbb{R}^+$ with $\sum_{x \in B(w, R)} g(x) \nu_x \leq 1$. Then*

$$\sum_{x,y \in B(w, R)} (g(x) - g(y))^2 \mu_{xy} \geq c_1 r^{-2} \left(\sum_{x \in B(w, R)} g(x)^2 \nu_x - \frac{c_2}{V(w, R)} \left(\frac{\rho_w(R)}{r} \right)^{c_3} \right). \quad (2.12)$$

Proof. Let $\hat{r} = \min\{\rho_x^{-1}(r) : x \in B(w, R)\} \wedge R$. Since $r \leq \rho_w(R)$ and $\rho_x(R) \asymp \rho_w(R)$ for each $x \in B(w, R)$, we have

$$\rho_x(\hat{r}) \leq c_1 r. \quad (2.13)$$

Note that for any $x \in B(w, R)$,

$$\frac{\rho_x^{-1}(r)}{R} = \frac{\rho_x^{-1}(r)}{\rho_x^{-1}(\rho_x(R))} \geq c_1 \left(\frac{r}{\rho_x(R)} \right)^{c_1} \geq c_2 \left(\frac{r}{\rho_w(R)} \right)^{c_2}.$$

So, $\frac{\hat{r}}{R} \geq c_2 \left(\frac{r}{\rho_w(R)} \right)^{c_2}$. Using Lemma 2.5, we then have

$$\frac{V(w, R)}{V(x, \hat{r})} \leq c \left(\frac{R}{\hat{r}} \right)^c \leq c_3 \left(\frac{\rho_w(R)}{r} \right)^{c_3}. \quad (2.14)$$

Choose $B_i = B(x_i, r_i)$, $i = 1, \dots, N$ such that $B(w, R) = \cup_{i=1}^N B(x_i, r_i)$ and $\hat{r} \leq r_i \leq 2\hat{r}$ for each i , and

$$|\{i : x \in B(x_i, r_i)\}| \leq c_4 \quad \text{for all } x \in B(w, R). \quad (2.15)$$

Use Lemmas 2.6,

$$\begin{aligned}
\sum_{x,y \in \mathcal{B}} (g(x) - g(y))^2 \mu_{xy} &\geq c_4^{-1} \sum_{i=1}^N \sum_{x,y \in B_i} (g(x) - g(y))^2 \mu_{xy} \\
&\geq c_4^{-1} \sum_{i=1}^N [\rho_{x_i}(\hat{r})]^{-2} \sum_{x \in B_i} (g(x) - \bar{g}_i)^2 \nu_x \\
&\geq c_4^{-1} \sum_{i=1}^N (c_1 r)^{-2} \left(\sum_{x \in B_i} g(x)^2 \nu_x - \frac{(\sum_{x \in B_i} g(x) \nu_x)^2}{V(x_i, \hat{r})} \right) \\
&\geq (c_4 c_1^2)^{-1} r^{-2} \left(\sum_{x \in \mathcal{B}} g(x)^2 \nu_x - \frac{c_3}{V(w, R)} \left(\frac{\rho_w(R)}{r} \right)^{c_3} \sum_{i=1}^N \left(\sum_{x \in B_i} g(x) \nu_x \right)^2 \right),
\end{aligned}$$

where $\mathcal{B} = B(w, R)$ and \bar{g}_i is the mean of g on B_i . Using (2.15), we get

$$\sum_{i=1}^N \sum_{x \in B_i} g(x) \nu_x \leq c \sum_{x \in \mathcal{B}} g(x) \nu_x \leq c.$$

Combining these inequalities with $\sum_i a_i^2 \leq (\sum_i a_i)^2$ for all $a_i \geq 0$, we complete the proof. \square

Remark 2.8 *One cannot expect to improve Lemma 2.7 to the whole space such as*

$$\sum_{x, y \in \mathbb{Z}^d} (g(x) - g(y))^2 \mu_{xy} \geq c_1 r^{-2} \left(\sum_{x \in \mathbb{Z}^d} g(x)^2 \nu_x - \frac{c_2}{V(w, R)} \left(\frac{\rho_w(R)}{r} \right)^{c_3} \right) \quad (2.16)$$

for all $r \in (0, \rho_w(R)]$, and $g : \mathbb{Z}^d \rightarrow \mathbb{R}^+$ with $\sum_{x \in \mathbb{Z}^d} g(x) \nu_x \leq 1$.

To see this, we fix $\alpha \in (-1, 0)$ and $d \geq 2$. On the one hand, choose $R \geq 1$ and $w \in \mathbb{Z}^d$ with $|w| = R^{-\alpha^{-1}}$. Then $\rho_w(R) = 1$, and hence one can take $r = 1$ further. Such,

$$V(w, R) \asymp R^{d-1} \rho_w(R) = R^{d-1} \rightarrow \infty. \quad (2.17)$$

On the other hand, let $s \geq 1$, and take

$$g(x) = A(s - |x|) 1_{B(0, s)}(x), \quad x \in \mathbb{Z}^d,$$

where A is the constant which such that $\sum_x g(x) \nu_x = 1$. Then

$$\sum_{x, y \in \mathbb{Z}^d} (g(x) - g(y))^2 \mu_{xy} \leq A^2 \sum_{x, y \in B(0, s)} \mu_{xy} \leq c A^2 s^{d-\alpha},$$

and

$$\sum_{x \in \mathbb{Z}^d} g(x)^2 \nu_x \geq \frac{A^2 s^2}{4} \sum_{x \in B(0, s/2)} \nu_x \geq c A^2 s^{d+2+\alpha}.$$

So, as s goes to infinity,

$$\sum_{x, y \in \mathbb{Z}^d} (g(x) - g(y))^2 \mu_{xy} \ll \sum_{x \in \mathbb{Z}^d} g(x)^2 \nu_x. \quad (2.18)$$

By (2.18) and (2.17), the inequality (2.16) fails.

3 On-diagonal upper bound estimates

Fix $w \in \mathbb{Z}^d$, $R \geq 1$ and $T = \rho_w(R)^2$. In this section, our aim is to give an upper bound of $p_T(w, w)$. As Lemma 2.7 and Remark 2.8 say, we have a good ball $B(w, R)$ only. So, we turn to the random walk X with reflection at $\partial_i B(w, R)$. By the approach of Barlow and

Chen [4], we obtain upper bounds of the heat kernel of the reflection process, and then bring these bounds back to the original process.

Write $\mathcal{B} = B(w, R)$ for short. Let Y be the continuous time random walk on \mathcal{B} with generator

$$\mathcal{L}_{\mathcal{B}}f(x) = \frac{1}{\nu_x} \sum_{y \in \mathcal{B}} (f(y) - f(x))\mu_{xy}.$$

For $x \in \mathbb{Z}^d$ and $r > 0$, set

$$\tau_{x,r} = \inf\{t \geq 0 : X_t \notin B(x, r)\}. \quad (3.1)$$

If Y and X start at the same vertex in $B(w, R-1)$, then we can couple Y and X on the same probability space such that

$$Y_s = X_s \quad \text{for } 0 \leq s \leq \tau_{w,R-1}. \quad (3.2)$$

We use \mathbb{P}_x for both X and Y . Denote the heat kernel of Y by

$$q_t(x, y) = \frac{\mathbb{P}_x(Y_t = y)}{\nu_y}.$$

Proposition 3.1 *For $u \in \mathcal{B}$ and $t \in (0, T]$,*

$$q_t(u, u) \leq \frac{c_1}{V(w, R)} \left(\frac{T}{t}\right)^{c_2}. \quad (3.3)$$

Epecially, $q_T(w, w) \leq \frac{c_1}{V(w, R)}$.

Proof. Given Lemma 2.7, the proof is similar to [2, Proposition 3.1] and [4, Proposition 3.2], so we omit it. \square

Lemma 3.2 *Let $x_1, x_2 \in \mathcal{B}$ with $|x_1 - x_2| \geq \frac{1}{16}R$. If $t \leq c_1T$ and $R \geq c_2$, then*

$$q_t(x_1, x_2) \leq \frac{1}{4V(w, R)}. \quad (3.4)$$

Proof. Write $\eta = \max_{x \in \mathcal{B}} \nu_x$. By (2.5) and (2.6), we have

$$\frac{T^{1/2}}{\eta} = \frac{\rho_w(R)}{\max_{x \in \mathcal{B}} \rho_x(1)} = \inf_{x \in \mathcal{B}} \left\{ \frac{\rho_w(R)}{\rho_x(R)} \cdot \frac{\rho_x(R)}{\rho_x(1)} \right\} \geq c_1 R^{c_1}. \quad (3.5)$$

Set $c_2 = 2^{|\alpha|+2}d$. Let $\tilde{\nu}_x = \eta^{-1}\nu_x$, $\tilde{\mu}_{xy} = \eta\mu_{xy}$ and $\tilde{\rho}(x, y) = c_2^{-1}\eta^{-1}\rho(x, y)$ for $x, y \in \mathcal{B}$. Then

$$\begin{cases} \frac{1}{\tilde{\nu}_x} \sum_{y \in \mathcal{B}} \tilde{\rho}(x, y)^2 \tilde{\mu}_{xy} \leq 1; \\ \tilde{\rho}(x, y) \leq 1 \quad \text{whenever } x \sim y. \end{cases} \quad (3.6)$$

Hence $\tilde{\rho}(\cdot, \cdot)$ is an adapted metric, which was introduced by Davies [20] and [21]. Let $Z_s = Y_{\eta^2 s}$, for $s \geq 0$. Then Z has the generator

$$\widetilde{\mathcal{L}}_B f(x) = \frac{1}{\tilde{\nu}_x} \sum_{y \in \mathcal{B}} (f(y) - f(x)) \tilde{\mu}_{xy}.$$

We state that there exists constant $c, c' > 0$ such that if $s \leq c\eta^{-2}T$ and $R \geq c'$ then

$$\mathbb{P}_{x_1}(Z_s = x_2) \leq \frac{\nu_{x_2}}{4V(w, R)}. \quad (3.7)$$

If this is true, then we have (3.4) and prove the lemma.

We now prove (3.7). Set $c_3 = c_{3.1.2} + c_1^{-1}c_{2.5.1}$. For each $i \in \{1, 2\}$, define

$$f_{x_i}(s) = \frac{V(w, R)}{c_{3.1.1}\nu_{x_i}} \left(\frac{\eta^2 s}{T} \right)^{c_3}, \quad s \geq 0.$$

Then by Proposition 3.1, for $s \leq \eta^{-2}T$,

$$\mathbb{P}_{x_i}(Z_s = x_i) = \mathbb{P}_{x_i}(Y_{\eta^2 s} = x_i) = q_{\eta^2 s}(x_i, x_i)\nu_{x_i} \leq \frac{1}{f_{x_i}(s)}. \quad (3.8)$$

Next we shall estimate the off-diagonal transition probability $\mathbb{P}_{x_1}(Z_s = x_2)$ by using the 'two-point' method of Grigor'yan-see [15, 11, 13, 8]. The metric $d_\nu(x, y)$ in [8] is just $\tilde{\rho}(x, y)$ and one can easily check that $f_{x_i}(s)$ is $(1, 2)$ -regular on $(0, T]$: see [15, 8] for the definition. By (3.5) and Lemma 2.5, for $s \leq \eta^{-2}T$,

$$\begin{aligned} \frac{f_{x_i}(s)}{s^{c_3}} &= \frac{V(w, R)}{c_{3.1.1}\nu_{x_i}} \cdot \left(\frac{\eta^2}{T} \right)^{c_3} \leq \frac{c_{2.5.1}R^{c_{2.5.1}}}{c_{3.1.1}} \cdot (c_1 R^{c_1})^{-2c_3} \\ &= c' R^{c_{2.5.1} - 2c_1 c_3} \leq c' R^{-c_{2.5.1}} \leq c'. \end{aligned}$$

Therefore, by [8, Theorem 1.1] for $s \in (\tilde{\rho}(x_1, x_2), \eta^{-2}T]$,

$$\mathbb{P}_{x_1}(Z_s = x_2) \leq \frac{c_4(\tilde{\nu}_{x_2}/\tilde{\nu}_{x_1})^{1/2}}{\sqrt{f_{x_1}(c_5 s)f_{x_2}(c_5 s)}} \exp \left(-c_6 \frac{\tilde{\rho}(x_1, x_2)^2}{s} \right) \quad (3.9)$$

$$= \frac{c_7 \nu_{x_2}}{V(w, R)} \left(\frac{T}{\eta^2 s} \right)^{c_3} \exp \left(-c_6 c_2^{-2} \frac{\rho(x_1, x_2)^2}{\eta^2 s} \right). \quad (3.10)$$

By Lemma 2.2 and the condition $|x_1 - x_2| \geq \frac{1}{16}R$, we have

$$\rho(x_1, x_2) \geq c_{2.2.2}\rho_{x_1}(\frac{1}{16}R) \geq c_8 \rho_w(R) = c_8 T^{1/2}. \quad (3.11)$$

Substituting (3.11) into (3.10) gives

$$\mathbb{P}_{x_1}(Z_s = x_2) \leq \frac{c_7 \nu_{x_2}}{V(w, R)} \left(\frac{T}{\eta^2 s} \right)^{c_3} \exp \left(-c_6 c_2^{-2} c_8^2 \frac{T}{\eta^2 s} \right),$$

which implies (3.7) holds for each $s \in (\tilde{\rho}(x_1, x_2), c_9 \eta^{-2} T]$, provided $c_9 > 0$ is small enough.

On the other hand, by [8, Corollary 2.8] we have the ‘long range’ bounds, that is, if $s \leq \tilde{\rho}(x_1, x_2)$ then

$$\mathbb{P}_{x_1}(Z_s = x_2) \leq c'(\tilde{\nu}_{x_2}/\tilde{\nu}_{x_1})^{1/2} e^{-c\tilde{\rho}(x_1, x_2)}. \quad (3.12)$$

Using (3.5) and (3.11), we have

$$\tilde{\rho}(x_1, x_2) = c_2^{-1} \eta^{-1} \rho(x_1, x_2) \geq c \eta^{-1} T^{1/2} \geq c' R^{c''}. \quad (3.13)$$

Combining these inequalities with Lemma 2.5,

$$\begin{aligned} \mathbb{P}_{x_1}(Z_s = x_2) &\leq c(\nu_{x_2}/\nu_{x_1})^{1/2} e^{-c' R^{c''}} = \frac{c\nu_{x_2}}{V(w, R)} \cdot \frac{V(w, R)}{(\nu_{x_2}\nu_{x_1})^{1/2}} e^{-c' R^{c''}} \\ &\leq \frac{c\nu_{x_2}}{V(w, R)} \cdot c_{2.5.1} R^{c_{2.5.1}} e^{-c' R^{c''}}. \end{aligned}$$

So, (3.7) holds again if $s \leq \tilde{\rho}(x_1, x_2)$ and $R \geq c$. \square

Lemma 3.3 *Let $t \leq c_1 T$ and $x \in B(w, \frac{7}{8}R)$. If $R \geq c_2$ then*

$$\mathbb{P}_x(Y_t \notin B(x, \frac{1}{16}R)) \leq \frac{1}{4}.$$

Proof. By Lemma 3.2, we get

$$\mathbb{P}_x(Y_t \notin B(x, \frac{1}{16}R)) = \sum_{y \in \mathcal{B} - B(x, \frac{1}{16}R)} q_t(x, y) \nu_y \leq \sum_{y \in \mathcal{B} - B(x, \frac{1}{16}R)} \frac{\nu_y}{4V(w, R)} \leq \frac{1}{4}.$$

\square

Now we bring these bounds of the reflection process back to the original process. Note that X and Y agree until time $\tau_{w, R-1}$.

Lemma 3.4 *If $R \geq c_1$ then for $x \in B(w, \frac{5}{8}R)$,*

$$\mathbb{P}_x(\tau_{x, R/8} < c_2 T) \leq \frac{1}{2}.$$

Proof. Given Lemma 3.3, the proof is similar to [4, Lemma 4.1], so we omit it. \square

Proposition 3.5 *Let $w \in \mathbb{Z}^d$, $R > 0$ and $T = \rho_w(R)^2$. Then*

$$\mathbb{P}_w(X_T = w) \leq \frac{c_1 \nu_w}{V(w, R)}.$$

Proof. If $R < (c_{3.4.1} \vee c_{3.2.2})$ then by Lemma 2.5,

$$\frac{\nu_w}{V(w, R)} \geq c' R^{-c} \geq c' (c_{3.4.1} \vee c_{3.2.2})^{-c} \geq c_1^{-1} \mathbb{P}_w(X_T = w).$$

So, let $R \geq (c_{3.4.1} \vee c_{3.2.2})$. Given Lemma 3.4, similar to the inequality (4.6) of Barlow and Chen [4] we obtain

$$p_{c_2 T}(w, w) \leq q_{c_2 T}(w, w) + \sup_{0 < s \leq c_2 T} \max_{y \in A} q_s(y, w),$$

where $c_2 = c_{3.4.1} \wedge c_{3.2.1} \wedge 1$ and $A = B(w, 5R/8) - B(w, 5R/8 - 1)$. By Proposition 3.1 and Lemma 3.2,

$$p_T(w, w) \leq p_{c_2 T}(w, w) \leq \frac{c_3}{V(w, R)}.$$

□

4 Near diagonal lower bound estimates

In this section, we shall prove the following lower bounds for the near diagonal transition probabilities. Recall $\tau_{x,r}$ from section 3. Fix $\delta \in (0, 1/2)$. We will use the notation K_i to denote constants which depend only δ, α and d , while $c_i = c_i(\alpha, d)$ as before.

Theorem 4.1 *Let $w \in \mathbb{R}^d$ and $R \geq 1$. For $x_1, x_2 \in B(w, R)$ and $t \in [\delta \rho_w(R)^2, 2\rho_w(R)^2]$,*

$$\mathbb{P}_{x_1}(X_t = x_2, \tau_{w, c_1 R} > t) \geq K_2 \frac{\nu_{x_2}}{V(w, R)}. \quad (4.1)$$

Since $\mu(B(w, R))$ do not satisfy the volume doubling property, we cannot obtain the lower bound by a general approach. Let us begin with a ball far from the origin.

Lemma 4.2 *Let $w \in \mathbb{Z}^d$ and $R \geq 1$ with $|w| \geq 32R$. Then for any $x_1, x_2 \in B(w, R)$ and $t \in [\delta \rho_w(R)^2, 2\rho_w(R)^2]$,*

$$\mathbb{P}_{x_1}(X_t = x_2, \tau_{w, 8R} > t) \geq K_1 R^{-d}. \quad (4.2)$$

Proof. Since $|w| \geq 32R$, $\rho_w(R) = R|w|^\alpha$, moreover, for any $x, y \in B(w, 16R)$ with $x \sim y$,

$$\nu_x \in [4dc_1^{-1}|w|^\alpha, c_1|w|^\alpha] \quad \text{and} \quad \mu_{xy} \in [c_1^{-1}|w|^{-\alpha}, c_1|w|^{-\alpha}].$$

By the application of Lemma 3.4 on $B(w, 8R)$, there exists $c_2 \in (0, 1/2)$ such that

$$\mathbb{P}_x(\tau_{x, R} > c_2 \rho_w(R)^2) \geq \frac{1}{2}, \quad \text{for all } x \in B(w, R). \quad (4.3)$$

For each $x, y \in B(w, 16R)$, we set

$$\tilde{\nu}_x = c_1|w|^{-\alpha}\nu_x \quad \text{and} \quad \tilde{\mu}_{xy} = \begin{cases} c_1^{-1}|w|^\alpha\mu_{xy}, & \text{if } x \neq y; \\ \tilde{\nu}_x - c_1^{-1}|w|^\alpha \sum_{z \in B(w, 16R) \setminus \{x\}} \mu_{xz}, & \text{if } x = y. \end{cases}$$

So, $\tilde{\nu}_x, \tilde{\mu}_{xy} \in [c_3, c_3^{-1}]$ for all $x \in B(w, 16R)$ and $y \in B(w, 16R) \cap B(x, 1)$. Let Z be the continuous time (constant speed) random walk on $B(w, 16R)$ with generator

$$\widetilde{\mathcal{L}}f(u) = \frac{1}{\tilde{\nu}_u} \sum_{v \in B(w, 16R)} (f(v) - f(u))\tilde{\mu}_{uv}.$$

Then Z and X can be coupled in the same probability such that

$$Z_s = X_{c_1^{-2}|w|^{2\alpha}s}, \quad \text{for all } s < \sigma = c_1^2|w|^{-2\alpha}\tau_{w, 8R},$$

where $\sigma := \inf\{s \geq 0 : Z_s \notin B(w, 8R)\}$. Fix $x_1, x_2 \in B(w, R)$, and let $u(s, y) = \mathbb{P}_{x_1}(Z_s = y, \sigma > s)/\tilde{\nu}_y$ for each $y \in B(w, 16R)$ and $s \geq 0$. Then u is a positive solution of the heat equation $\frac{\partial u}{\partial s} = \widetilde{\mathcal{L}}u$ on $(0, \infty) \times B(w, 4R)$. One can easily check that $DV(C_1), P(C_2)$ and $\Delta(\alpha)$ hold for the weighted graph with vertex set $B(w, 16R)$ and edge weight $\tilde{\mu}_{xy}$, and so $u(s, y)$ satisfies the Harnack inequality, see [12, Theorem 1.7]. Therefore,

$$\max_{[\frac{1}{2}s_0, s_0] \times B(w, 2R)} u \leq K_1^{-1} \min_{[\delta c_1^2 R^2, 2c_1^2 R^2] \times B(w, 2R)} u,$$

where $s_0 = \delta c_1^2 R^2$. Furthermore, for any $s \in [\delta c_1^2 R^2, 2c_1^2 R^2]$,

$$\begin{aligned} \mathbb{P}_{x_1}(Z_s = x_2, \sigma > s) &\geq K_1 \left(\sum_{z \in B(w, 2R)} \tilde{\nu}_z \right)^{-1} \sum_{z \in B(w, 2R)} \mathbb{P}_{x_1}(Z_{s_0} = z, \sigma > s_0) \\ &\geq K_1 c_3 |B(w, 2R)|^{-1} \mathbb{P}_{x_1}(Z_{s_0} \in B(w, 2R), \sigma > s_0) \\ &\geq K_1 c_3 (5R)^{-d} \mathbb{P}_{x_1}(\inf\{h : Z_h \notin B(x_1, R)\} > s_0). \end{aligned} \quad (4.4)$$

Since $X_t = Z_{c_1^2|w|^{-2\alpha}t}$ for all $t < \tau$, inequality (4.4) can be rewrote as

$$\mathbb{P}_{x_1}(X_t = x_2, \tau_{w, 8R} > t) \geq K_2 R^{-d} \mathbb{P}_{x_1}(\tau_{x_1, R} > \delta c_2 R^2 |w|^{2\alpha}), \quad t \in [\delta R^2 |w|^{2\alpha}, 2R^2 |w|^{2\alpha}].$$

Using (4.3), we finish the proof. \square

Lemma 4.3 For any $t \in [\delta R^{2+2\alpha}, R^{2+2\alpha}]$ and $x \in B(0, R)$,

$$\mathbb{P}_x(|X_t| > K_1 R, \tau_{x, c_2 R} > t) \geq \frac{1}{4}.$$

Proof. By Proposition 3.5, for any $x, y \in \mathbb{Z}$ and $t > 0$,

$$p_t(x, y) \leq (p_t(x, x)p_t(y, y))^{1/2} \leq c_1 (V(x, \rho_x^{-1}(t^{1/2}))V(y, \rho_y^{-1}(t^{1/2})))^{-1/2}.$$

So, from Lemma 2.4 we can get, if $x, y \in B(0, R)$ and $t \in [\delta R^{2+2\alpha}, R^{2+2\alpha}]$ then

$$p_t(x, y) \leq K_1 t^{-(d+\alpha)/(2+2\alpha)} \leq K_2 R^{-d-\alpha}.$$

Fix $x \in B(0, R)$ and $t \in [\delta R^{2+2\alpha}, R^{2+2\alpha}]$. By Lemma 2.4 again, for each $\varepsilon \in (0, 1)$,

$$\begin{aligned}\mathbb{P}_x(|X_t| \leq \varepsilon R) &= \sum_{y \in B(0, \varepsilon R)} p_t(x, y) \nu_y \leq V(0, \varepsilon R) \cdot K_2 R^{-(d+\alpha)} \\ &\leq c_2(\varepsilon R)^{d+\alpha} \cdot K_2 R^{-(d+\alpha)} = K_2 c_2 \varepsilon^{d+\alpha}.\end{aligned}$$

Hence there exists $\varepsilon_0 = \varepsilon_0(\delta, \alpha, d) > 0$ such that

$$\mathbb{P}_x(|X_t| \leq \varepsilon_0 R) \leq \frac{1}{4}. \quad (4.5)$$

On the other hand, applying Lemma 3.4 gives

$$\mathbb{P}_x(\tau_{x, cR} < t) \leq \mathbb{P}_x(\tau_{x, cR} < R^{2+2\alpha}) \leq \frac{1}{2}. \quad (4.6)$$

Combing (4.6) with (4.5), we finish the proof. \square

Lemma 4.4 *Let $R \geq 1$. Let $x_1, x_2 \in B(0, R) \setminus B(0, \delta R)$ and $t \in [\delta R^{2+2\alpha}, R^{2+2\alpha}]$. Then*

$$\mathbb{P}_{x_1}(X_t = x_2, \tau_{0, 10R} > t) \geq K_1 R^{-d}. \quad (4.7)$$

Proof. Write $\mathbb{T} = B(0, R) \setminus B(0, \delta R)$ for short. If $d \geq 2$, then \mathbb{T} is connected. Note that $\rho_w(R) \subset [K_1^{-1}R^{1+\alpha}, K_1R^{1+\alpha}]$ and $B(w, 8R) \subset B(0, 10R)$ for all $w \in \mathbb{T}$, and there exist vertices $w_i \in \mathbb{T}$, $i \leq K_2$ such that $\mathbb{T} = \cup_{i=1}^{K_2} B(w_i, \frac{\delta}{64}R)$. A standard chaining argument using Lemma 4.2 on $B(w_i, \frac{\delta}{32}R)$, proves (4.7) for $d \geq 2$. Next, we consider $d = 1$. Since $\mathbb{T} = ([-R, -\delta R] \cup [\delta R, R]) \cap \mathbb{Z}$ is not connected, we have to discuss the problem on several cases.

Case I: $x_1, x_2 > 0$. Then x_1 and x_2 can be joint with a sequence of balls $B(w_i, \frac{\delta}{32}R)$ within $[\delta R, R] \cap \mathbb{Z}$ as before. Hence (4.7) holds for this case, too.

Case II: $x_1 > 0 > x_2$. For conciseness, we write $\widehat{\mathbb{P}}$ for the measure of the process X killed on exiting $B(0, 10R)$. Let $\varepsilon_0 = \varepsilon_0(\delta, \alpha, d) \in (0, 1)$ be a small constant, whose value will be taken later. Set $x_* = \lfloor \varepsilon_0 R \rfloor$. By the result of Case I, we have

$$\inf_{s \in [\delta' R^{2+2\alpha}, R^{2+2\alpha}]} \inf_{x, y \in B(0, R) \setminus B(0, \delta') \cap \mathbb{Z}} \widehat{\mathbb{P}}_x(X_s = y) \geq K_3 R^{-1},$$

where $\delta' = \min\{\frac{1}{2}\varepsilon_0, \frac{1}{3}\delta\}$. So,

$$\widehat{\mathbb{P}}_{x_1}(X_{t/3} \in (\frac{1}{2}\varepsilon_0 R, \varepsilon_0 R)) \geq \frac{1}{4} K_3 \varepsilon_0 \quad \text{and} \quad \inf_{s \in [t/3, t]} \widehat{\mathbb{P}}_{-x_*}(X_s = x_2) \geq K_3 R^{-1}.$$

For $x \in \mathbb{Z}$, we define $\sigma_x = \inf\{t \geq 0 : X_t = x\}$, the first time of visiting vertex x . By the strong Markov property,

$$\begin{aligned}\mathbb{P}_{x_1}(X_t = x_2, \tau_{0, 10R} > t) &\geq \widehat{\mathbb{P}}_{x_1}(\sigma_{x_*} < \frac{t}{3}, \sigma_{-x_*} < \frac{2t}{3}, X_t = x_2) \\ &\geq \widehat{\mathbb{P}}_{x_1}(\sigma_{x_*} < \frac{t}{3}) \widehat{\mathbb{P}}_{x_*}(\sigma_{-x_*} < \frac{t}{3}) \inf_{s \in [t/3, t]} \widehat{\mathbb{P}}_{-x_*}(X_s = x_2) \\ &\geq \widehat{\mathbb{P}}_{x_1}(X_{t/3} \in (\frac{1}{2}\varepsilon_0 R, \varepsilon_0 R)) \widehat{\mathbb{P}}_{x_*}(\sigma_{-x_*} < \frac{t}{3}) \inf_{s \in [t/3, t]} \widehat{\mathbb{P}}_{-x_*}(X_s = x_2)\end{aligned}$$

$$\geq \frac{1}{4} K_3 \varepsilon_0 \cdot \widehat{\mathbb{P}}_{x_*}(\sigma_{-x_*} < \frac{t}{3}) \cdot K_3 R^{-1}. \quad (4.8)$$

So, we need a lower bound of $\widehat{\mathbb{P}}_{x_*}(\sigma_{-x_*} < \frac{t}{3})$. By Lemma 2.2, for any $x \in \mathbb{N}$ and $r, s \geq 2|x|$,

$$\mathbb{P}_x(\sigma_{x-r} > \sigma_{x+s}) = \frac{\sum_{i=x-r}^{x-1} \mu_{i,i+1}^{-1}}{\sum_{i=x-r}^{x+s-1} \mu_{i,i+1}^{-1}} \leq \frac{c_{2.2.1}^{-1} \rho_x(r)}{c_{2.2.1} \rho_x(r+s-1)} \leq c_1 \left(\frac{r}{r+s} \right)^\alpha. \quad (4.9)$$

So, there exists $c_2 \in \mathbb{N}$ such that

$$\mathbb{P}_x(\sigma_{-x} > \sigma_{c_2 x}) \leq \frac{1}{8}, \quad \text{for all } x \in \mathbb{N}. \quad (4.10)$$

By Lemma 4.3, there exist $c_3 = c_3(\alpha, d) \in (0, 1)$ and $K_4 = K_4(\delta, \alpha, d) \in (0, 1)$ such that

$$\mathbb{P}_x(|X_{t/3}| > K_4 R, \tau_{x,R} > t/3) \geq \frac{1}{4}, \quad x \in B(0, c_3 R).$$

Now we choose $\varepsilon_0 = c_2^{-1} K_4 c_3$. Then $x_* = \lfloor c_2^{-1} K_4 c_3 R \rfloor \in B(0, c_3 R)$ and so,

$$\mathbb{P}_{x_*}(\sigma_{-x_*} \wedge \sigma_{c_2 x_*} < \frac{1}{3}t, \tau_{0,10R} > \frac{1}{3}t) \geq \mathbb{P}_{x_*}(|X_{t/3}| > K_4 R, \tau_{c_2 x_*, R} > t/3) \geq \frac{1}{4}. \quad (4.11)$$

Combining (4.11) with (4.10), we get

$$\widehat{\mathbb{P}}_{x_*}(\sigma_{-x_*} < \frac{t}{3}) \geq \frac{1}{8}.$$

Substituting the above inequality into (4.8), we prove (4.7) for the second case.

By symmetry, we have (4.7) as $x_1 < 0$. Therefore, (4.7) holds in any case. \square

Proof of Theorem 4.1. If $|w| \geq 32R$, then one can take $c_1 = 8$ in (4.1) and the problem is reduced to Lemma 4.2. So, let $R > |w|/32$ in the following. Then

$$\rho_w(R) \in [c_1 R^{1+\alpha}, c_2 R^{1+\alpha}] \quad \text{and} \quad B(w, R) \subset B(0, 40R).$$

Fix $t \in [\delta \rho_w(R)^2, 2\rho_w(R)^2]$. Then $t \in [c_1 \delta R^{2+2\alpha}, c_2 R^{2+2\alpha}]$. By Lemma 4.3, for any $x \in B(0, 40R)$,

$$\mathbb{P}_x(|X_{t/3}| > K_1 R, \tau_{0,c_3 R} > t/3) \geq \frac{1}{4}. \quad (4.12)$$

Write $\mathbb{T} = B(0, c_3 R) \setminus B(0, K_1 R)$. By Lemma 4.4, for all $x, y \in \mathbb{T}$,

$$\mathbb{P}_x(X_{t/3} = y, \tau_{0,10c_3 R} > t/3) \geq K_2 R^{-d}.$$

Therefore, for any $x_1, x_2 \in B(w, R) \subset B(0, 40R)$,

$$\begin{aligned} \mathbb{P}_{x_1}(X_t = x_2, \tau_{0,10c_3 R} > t) &\geq \sum_{x,y \in \mathbb{T}} \widehat{\mathbb{P}}_{x_1}(X_{t/3} = x, X_{2t/3} = y, X_t = x_2) \\ &= \sum_{x,y \in \mathbb{T}} \widehat{\mathbb{P}}_{x_1}(X_{t/3} = x) \widehat{\mathbb{P}}_x(X_{t/3} = y) \widehat{\mathbb{P}}_y(X_{t/3} = x_2) \\ &\geq K_2 R^{-d} \sum_{x,y \in \mathbb{T}} \widehat{\mathbb{P}}_{x_1}(X_{t/3} = x) \widehat{\mathbb{P}}_y(X_{t/3} = x_2) \end{aligned}$$

$$\begin{aligned}
&= K_2 R^{-d} \sum_{x,y \in \mathbb{T}} \widehat{\mathbb{P}}_{x_1}(X_{t/3} = x) \widehat{\mathbb{P}}_{x_2}(X_{t/3} = y) \frac{\nu_{x_2}}{\nu_y} \\
&\geq K_2 R^{-d} \frac{\nu_{x_2}}{\max_{y \in \mathbb{T}} \nu_y} \sum_{x,y \in \mathbb{T}} \widehat{\mathbb{P}}_{x_1}(X_{t/3} = x) \widehat{\mathbb{P}}_{x_2}(X_{t/3} = y) \\
&\geq K_2 R^{-d} \frac{\nu_{x_2}}{K_3 R^\alpha} \widehat{\mathbb{P}}_{x_1}(|X_{t/3}| \in \mathbb{T}) \widehat{\mathbb{P}}_{x_2}(|X_{t/3}| \in \mathbb{T}) \\
&\geq \frac{K_2}{16 K_3} \frac{\nu_{x_2}}{R^{d+\alpha}}, \tag{4.13}
\end{aligned}$$

where we use $\widehat{\mathbb{P}}$ to denote the measure of the process X killed on exiting $B(0, 10c_3 R)$. Substituting $V(w, R) \leq cR^{d+\alpha}$ and $\tau_{0,10c_3 R} \leq \tau_{w,cR}$ into (4.13), we complete the proof. \square

5 Proof of Theorem 1.1

Lemma 5.1 *There exists constant $c_1 > 0$ such that for any $x, y \in \mathbb{Z}^d$,*

$$(\nu_x \vee \nu_y) |\log \nu_x - \log \nu_y|^3 \leq c_1 \rho(x, y).$$

Proof. Let $|x| > |y| \geq 1$. Directly calculate

$$\begin{aligned}
&\frac{(\nu_x \vee \nu_y)}{(|x| \vee |y|)^\alpha} \cdot \frac{|\log \nu_x - \log \nu_y|^3}{|x - y|} = \frac{|x|^\alpha \vee |y|^\alpha}{|x|^\alpha} \cdot \frac{|\log(|x|^\alpha) - \log(|y|^\alpha)|^3}{|x - y|} \\
&= \left(\frac{|y|}{|x|}\right)^{\alpha \wedge 0} \cdot |\alpha|^3 \frac{\log^3(|x|/|y|)}{(|x|/|y| - 1)|y|} \\
&\leq |\alpha|^3 \sup_{t > 1} \left\{ t^{(-\alpha) \vee 0} \cdot \frac{\log^3 t}{t - 1} \right\}.
\end{aligned}$$

Since $\alpha > -1$, the supremum of the right side is finite and hence if $|x| > |y| \geq 1$ then

$$(\nu_x \vee \nu_y) |\log \nu_x - \log \nu_y|^3 \leq c(|x| \vee |y|)^\alpha |x - y| \leq c' \rho(x, y).$$

The proof of the rest case is the same and so we omit the details. \square

Proof of Theorem 1.1. We obtain the Gaussian upper bounds by the same way as Lemma 3.2. Write $\eta = \nu_x \vee \nu_y$ for short. Set $\tilde{\nu}_u = \eta^{-1} \nu_u$ and $\tilde{\mu}_{uv} = \eta \mu_{uv}$ for each $u, v \in \mathbb{Z}^d$. Denote $\tilde{\rho} : \mathbb{Z}^d \times \mathbb{Z}^d \rightarrow \mathbb{R}^+$ by

$$\tilde{\rho}(u, v) = ((2^{|\alpha|+2}d)^{-1} \cdot \eta^{-1} \rho(u, v)) \wedge |u - v|_1.$$

Then $\tilde{\rho}(\cdot, \cdot)$ is an adapted metric of \mathbb{Z}^d , that is, for all $u \in \mathbb{Z}^d$,

$$\begin{cases} \frac{1}{\tilde{\nu}_u} \sum_{v \in \mathbb{Z}^d} \tilde{\rho}(u, v)^2 \tilde{\mu}_{uv} \leq 1; \\ \tilde{\rho}(u, v) \leq 1 \quad \text{whenever } v \sim u. \end{cases}$$

By Lemma 2.2,

$$\eta^{-1}\rho(x, y) \leq (\nu_x \vee \nu_y)^{-1} \cdot c(|x| \vee |y|)^\alpha |x - y| \leq c|x - y|_1.$$

So,

$$c_1^{-1}\eta^{-1}\rho(x, y) \leq \tilde{\rho}(x, y) \leq c_1\eta^{-1}\rho(x, y). \quad (5.1)$$

Set $Z_s = X_{\eta^2 s}$ for $s \geq 0$. Then Z has the generator

$$\tilde{\mathcal{L}}f(u) = \frac{1}{\tilde{\nu}_u} \sum_{v \in \mathbb{Z}^d} (f(v) - f(u))\tilde{\mu}_{uv}.$$

By Proposition 3.5 for each $z \in \{x, y\}$,

$$\mathbb{P}_z(Z_s = z) = \mathbb{P}_z(X_{\eta^2 s} = z) \leq \frac{c_2\nu_z}{V(z, \rho_z^{-1}(\eta s^{1/2}))} := \frac{1}{f_z(s)}.$$

By Lemma 2.5 and the inequality (2.5), for each $s \geq (\log \nu_x - \log \nu_y)^2$ we have

$$f_z(s) \leq \frac{V(z, \rho_z^{-1}(\eta e^{s^{1/2}}))}{c_2\nu_z} \leq c(\rho_z^{-1}(\eta e^{s^{1/2}}))^c \leq c' \left(\frac{\eta e^{s^{1/2}}}{\rho_z(1)} \right)^{c'} \leq c' \left(\frac{\nu_x \vee \nu_y}{\nu_x \wedge \nu_y} e^{s^{1/2}} \right)^{c'} \leq c' e^{2c' s^{1/2}}.$$

Therefore, similar to (3.9) we can apply [8, Theorem 5.1] and get

$$\mathbb{P}_x(Z_s = y) \leq \frac{c_3(\nu_y/\nu_x)^{1/2}}{\sqrt{f_z(s/c_3)f_y(s/c_3)}} \exp \left(-\frac{\tilde{\rho}(x, y)^2}{c_3 s} \right) \quad \text{for all } s \geq |c_3 \log(\nu_x/\nu_y)|^3 \vee \tilde{\rho}(x, y).$$

By the inequality (5.1) and Lemma 5.1,

$$|c_3 \log(\nu_x/\nu_y)|^3 \leq c_4 \tilde{\rho}(x, y).$$

So, for each $t \geq c_4 c_1 \eta \rho(x, y)$, we have $\eta^{-2}t \geq |c_3 \log(\nu_x/\nu_y)|^3 \vee \tilde{\rho}(x, y)$ and

$$\begin{aligned} \mathbb{P}_x(X_t = y) &= \mathbb{P}_x(Z_{\eta^{-2}t} = y) \leq \frac{c_5(\nu_y/\nu_x)^{1/2}}{\sqrt{f_x(\eta^{-2}t/c_5)f_y(\eta^{-2}t/c_5)}} \exp \left(-\frac{\tilde{\rho}(x, y)^2}{c_5 \eta^{-2}t} \right) \\ &\leq \frac{c\nu_y}{\sqrt{V(x, \rho_x^{-1}(t^{1/2}))V(y, \rho_y^{-1}(t^{1/2}))}} \exp \left(-c' \frac{\rho(x, y)^2}{t} \right). \end{aligned} \quad (5.2)$$

Further, by Lemma 2.4 we conclude that

$$p_t(x, y) \leq \frac{c}{\sqrt{V_\rho(x, t^{1/2})V_\rho(y, t^{1/2})}} \exp \left(-c' \frac{\rho(x, y)^2}{t} \right), \quad t \geq c_4 c_1 \eta \rho(x, y). \quad (5.3)$$

On the other hand, by [8, Corollary 2.8], if $s \leq c_4 c_1^2 \tilde{\rho}(x, y)$ then

$$\mathbb{P}_x(Z_s = y) \leq c(\tilde{\nu}_y/\tilde{\nu}_x)^{1/2} \exp \left(-c' \tilde{\rho}(x, y) (1 \vee \log(\tilde{\rho}(x, y)/s)) \right).$$

Hence, for each $t \leq c_4 c_1 \eta \rho(x, y)$,

$$\begin{aligned} \mathbb{P}_x(X_t = y) &= \mathbb{P}_x(Z_{\eta^{-2}t} = y) \leq c(\tilde{\nu}_y/\tilde{\nu}_x)^{1/2} \exp\left(-c'\tilde{\rho}(x, y)(1 \vee \log(\eta^2\tilde{\rho}(x, y)/t))\right) \\ &\leq c(\nu_y/\nu_x)^{1/2} \exp\left(-c''\eta^{-1}\rho(x, y)(1 \vee \log(\eta\rho(x, y)/t))\right). \end{aligned} \quad (5.4)$$

Combining (5.4) with (5.3), we conclude that both (1.2) and (1.3) are true.

The Gaussian lower bound is proved by a standard chaining argument. If $t \geq \rho(x, y)^2$, then there exists $c_1 > 1$ such that $t \geq c_1^{-2}\rho_x(|x-y|)^2$. Applying Theorem 4.1 on $B(x, \rho_x^{-1}(c_1 t^{1/2}))$, we get

$$p_t(x, y) \geq \frac{c}{V(x, \rho_x^{-1}(c_1 t^{1/2}))} \geq \frac{c'}{V_\rho(x, t^{1/2})}. \quad (5.5)$$

So, let $(\nu_x \vee \nu_y)\rho(x, y) \leq t \leq \rho(x, y)^2$. Fix an L_1 -geodesic path γ from x to y . By Lemma 2.2, there exists $c_2 > 1$ such that

$$\nu(\gamma) \leq c_2 \rho(x, y).$$

Set $r = t/\rho(x, y)$, then

$$\rho(x, y) \geq r \geq \nu_x \vee \nu_y = \max_{z \in \gamma} \nu_z.$$

Hence there exists a sequence of vertices $y = z_0, z_1, \dots, z_m = x$ on the path γ , such that

$$m \leq 2c_2 \rho(x, y)/r = 2c_2 \frac{\rho(x, y)^2}{t} \quad \text{and} \quad r \leq \rho(z_{i-1}, z_i) \leq 2r \quad \text{for } i \leq m.$$

As a result,

$$|z_{i-1} - z_{i-2}| \leq c'_3 \rho_{z_{i-1}}^{-1}(\rho(z_{i-1}, z_{i-2})) \leq c'_3 \rho_{z_{i-1}}^{-1}(2\rho(z_{i-1}, z_i)) \leq (c_3 - 1)|z_i - z_{i-1}|.$$

Write $r_i = |z_i - z_{i-1}|$, $F_i = B(z_i, r_i)$ and $F_i^* = B(z_i, c_3 r_i)$ for $i \leq m$. Then

$$F_{i-1} \cup F_i \subset F_i^*.$$

Set $s = (4c_2)^{-1}r^2$. Then $s \asymp \rho(z_i, z_{i-1})^2 \asymp \rho_{z_i}(r_i)^2$. As (5.5), we have

$$p_s(y', x') \geq \frac{c_4}{\nu(F_i^*)} \quad \text{for } y' \in F_{i-1}, x' \in F_i. \quad (5.6)$$

By Lemma 2.5, for $y' \in F_{i-1}$,

$$\mathbb{P}_{y'}(X_s \in F_i) \geq c_4 \frac{\nu(F_i)}{\nu(F_i^*)} \geq c_5.$$

Note that

$$t - ms = t - m \cdot (4c_2)^{-1}r^2 \geq t - (2c_2 \rho(x, y)/r) \cdot (4c_2)^{-1}r \cdot (t/\rho(x, y)) = \frac{t}{2}.$$

So, as (5.5) we can get

$$p_{t-ms}(x, y') \geq \frac{c_6}{V_\rho(x, t^{1/2})} \quad \text{for } y' \in F_m. \quad (5.7)$$

Therefore,

$$\begin{aligned}
p_t(x, y) &= p_t(y, x) \geq \mu_x^{-1} \mathbb{P}_y(X_{is} \in F_i, 1 \leq i \leq m, X_t = x) \\
&\geq c_5^m \min_{y' \in F_m} \mathbb{P}_{y'}(X_{t-ms} = x) \mu_x^{-1} \\
&= c_5^m \min_{y' \in F_m} p_{t-ms}(x, y') \geq c_5^m \frac{c_6}{V_\rho(x, t^{1/2})} \\
&\geq \frac{c_6}{V_\rho(x, t^{1/2})} \exp\{-c'_5 m\} \geq \frac{c_6}{V_\rho(x, t^{1/2})} \exp\{-2c_2 c'_5 \frac{\rho(x, y)^2}{t}\},
\end{aligned}$$

which implies (1.4). We have completed the proof of Theorem 1.1. \square

6 Proof of Theorem 1.3

Proof of Theorem 1.3. (1) By Theorem 1.1 and Lemma 2.4, if $\alpha < d - 2$ then

$$\int_1^\infty p_t(0, 0) dt \leq c \int_1^\infty t^{-(d+\alpha)/(2+2\alpha)} dt = \frac{2+2\alpha}{d-2-\alpha} c < \infty.$$

Hence if $\alpha < d - 2$ then X is transient. Similarly, if $\alpha \geq d - 2$ then $\int_1^\infty p_t(0, 0) dt = \infty$ and so X is recurrent.

(2) Let X' be an independent copy of X . We use $\mathbb{P}_{x, x'}$ for the probability measure of the processes X and X' which start from x and x' respectively.

If $d = 1$ then

$$\begin{aligned}
\int_1^\infty \mathbb{P}_{0,0}(X_t = X'_t = 0) dt &= \int_1^\infty \mathbb{P}_0(X_t = 0) \mathbb{P}_0(X'_t = 0) dt \\
&= \int_1^\infty p_t(0, 0)^2 dt \geq c \int_1^\infty t^{-2(1+\alpha)/(2+2\alpha)} dt = \infty.
\end{aligned}$$

So, (X, X') is recurrent, which implies X and X' collide at the origin infinitely often.

Let $d = 2$. Fix $\lambda = \lceil 100c_{4,1,1} \rceil > 100$. For $k \geq 1$, we set

$$t_k = \lambda^{2k(1+\alpha)},$$

$$\mathbb{T}_k = B(0, 2\lambda^k) - B(0, \lambda^k),$$

$$\theta_k = \inf\{t \geq 0 : |X_t| \geq \lambda^{k+1}\}, \quad \theta'_k = \inf\{t \geq 0 : |X'_t| \geq \lambda^{k+1}\}$$

and

$$H_k = \int_0^{\theta_k \wedge \theta'_k \wedge 2t_k} 1_{\{X_t = X'_t \in \mathbb{T}_k\}} dt.$$

So, if $H_k > 0$ then there exists at least one collision of X and X' before their breaking out of $B(0, \lambda^{k+1})$. We shall use the second moment method to estimate the probability of the event $\{H_k > 0\}$ as the approach of [9, 10]. Fix $x, y \in B(0, \lambda^k)$. Then

$$\mathbf{E}_{x,y}(H_k) = \int_0^{2t_k} \mathbb{P}_{x,y}(X_t = X'_t \in \mathbb{T}_k, \theta_k > t, \theta'_k > t) dt$$

$$\begin{aligned}
&\geq \int_{t_k}^{2t_k} \sum_{u \in \mathbb{T}_k} \mathbb{P}_{x,y}(X_t = u, X'_t = u, \theta_k > t, \theta'_k > t) dt \\
&= \int_{t_k}^{2t_k} \sum_{u \in \mathbb{T}_k} \mathbb{P}_x(X_t = u, \theta_k > t) \mathbb{P}_y(X'_t = u, \theta'_k > t) dt.
\end{aligned} \tag{6.1}$$

Note that $t_k = 2^{-2-2\alpha} \rho_0 (2\lambda^k)^2$ and $\lambda^{k+1} = \lceil 100c_{4.1.1} \rceil \lambda^k$. Employing Theorem 4.1 on $B(0, 2\lambda^k)$, we get for each $u, v \in B(0, 2\lambda^k)$ and $t \in [t_k, 2t_k]$,

$$\mathbb{P}_u(X_t = v, \theta_k > t) \geq \frac{c\nu_v}{V(0, 2\lambda^k)}.$$

By Lemma 2.4, for $v \in \mathbb{T}_k$,

$$\frac{\nu_v}{V(0, 2\lambda^k)} \geq c \frac{|v|^\alpha}{(2\lambda^k)^{2+\alpha}} \geq c' \lambda^{-2k}.$$

Hence $\mathbb{P}_u(X_t = v, \theta_k > t) \geq c\lambda^{-2k}$ for each $u \in \{x, y\}$, $v \in \mathbb{T}_k$ and $t \in [t_k, 2t_k]$. Therefore, inequality (6.1) becomes

$$\mathbf{E}_{x,y}(H_k) \geq (c\lambda^{-2k})^2 \cdot |\mathbb{T}_k| \cdot t_k \geq c^2 \lambda^{-4k} \cdot c' (\lambda^k)^2 \cdot \lambda^{2k(1+\alpha)} = c'' \lambda^{2k\alpha}. \tag{6.2}$$

On the other hand, for any $u \in \mathbb{T}_k$,

$$\begin{aligned}
\mathbf{E}_{u,u}(H_k) &\leq \int_0^{2t_k} \sum_{w \in \mathbb{T}_k} [\mathbb{P}_u(X_t = w)]^2 dt \\
&\leq \frac{\max_{w \in \mathbb{T}_k} \nu_w}{\nu_u} \int_0^{2t_k} \sum_{w \in \mathbb{T}_k} \mathbb{P}_u(X_t = w) \mathbb{P}_w(X_t = u) dt \\
&\leq c \int_0^{2t_k} \mathbb{P}_u(X_{2t} = u) dt \leq c\nu_u^2 + c \int_{\nu_u^2}^{2t_k} \mathbb{P}_u(X_{2t} = u) dt \\
&\leq c\nu_u^2 + \int_{\nu_u^2}^{2t_k} \frac{c'\nu_u}{V_\rho(u, t^{1/2})} dt \\
&\leq c\nu_u^2 + c''\nu_u^2 \int_{\nu_u^2}^{2t_k} t^{-1} dt,
\end{aligned}$$

where the last second inequality is by (1.3), while the last by Lemma 2.4. Hence

$$\mathbf{E}_{u,u}(H_k) \leq c\nu_u^2 (1 + \log(2t_k) - \log(\nu_u^2)) \leq c' \lambda^{2k\alpha} \cdot (\log(\lambda^{2k(1+\alpha)}) - \log(\lambda^{2k\alpha})) = c'' k \lambda^{2k\alpha}. \tag{6.3}$$

By the strong Markov property,

$$\begin{aligned}
\mathbf{E}_{x,y}(H_k^2) &= 2\mathbf{E}_{x,y} \left(\int_0^{\theta_k \wedge \theta'_k \wedge 2t_k} 1_{\{X_t = X'_t \in \mathbb{T}_k\}} dt \int_t^{\theta_k \wedge \theta'_k \wedge 2t_k} 1_{\{X_s = X'_s \in \mathbb{T}_k\}} ds \right) \\
&\leq 2\mathbf{E}_{x,y} \left(\int_0^{\theta_k \wedge \theta'_k \wedge 2t_k} 1_{\{X_t = X'_t \in \mathbb{T}_k\}} \mathbf{E}_{X_t, X'_t}(H_k) dt \right)
\end{aligned}$$

$$\leq 2 \sup_{u \in \mathbb{T}_k} \mathbf{E}_{u,u}(H_k) \mathbf{E}_{x,y}(H_k).$$

So, by (6.3), (6.2) and the Cauchy Schwarz inequality,

$$\mathbb{P}_{x,y}(H_k > 0) \geq \frac{[\mathbf{E}_{x,y}(H_k)]^2}{\mathbf{E}_{x,y}(H_k^2)} \geq \frac{\mathbf{E}_{x,y}(H_k)}{2 \sup_{u \in \mathbb{T}_k} \mathbf{E}_{u,u}(H_k)} \geq \frac{c}{k}.$$

Therefore, when X and X' start from $x, y \in B(0, \lambda^k)$ respectively, the probability that they will collide before their breaking out $B(0, \lambda^{k+1})$, is not less than $\frac{c}{k}$. Note that $\sum_k \frac{1}{k} = \infty$. Using the second Borel-Cantelli Lemma as [10, Theorem 1.1], we prove that X and X' collide infinitely often when $d = 2$.

(3) Let $d \geq 3$. For $k \geq 0$, set

$$\mathbb{T}_k = B(0, 2^{k+1}) - B(0, 2^k) \quad \text{and} \quad Z_k = \int_0^\infty 1_{\{X_t = X'_t \in \mathbb{T}_k\}} dt.$$

Then

$$\begin{aligned} \mathbf{E}_{0,0}(Z_k) &= \sum_{u \in \mathbb{T}_k} \int_0^\infty [\mathbb{P}_0(X_t = u)]^2 dt \\ &= \sum_{u \in \mathbb{T}_k} \int_{t_k}^\infty [\mathbb{P}_0(X_t = u)]^2 dt + \sum_{u \in \mathbb{T}_k} \int_{s_k}^{t_k} [\mathbb{P}_0(X_t = u)]^2 dt + \sum_{u \in \mathbb{T}_k} \int_0^{s_k} [\mathbb{P}_0(X_t = u)]^2 dt \\ &= I_1 + I_2 + I_3, \end{aligned}$$

where $s_k = (1 \vee 2^{k\alpha})2^{k(1+\alpha)}$ and $t_k = 2^{k(2+2\alpha)}$. We shall deal with the three sums separately. Since $t_k \geq c\rho(0, u)^2$ for $u \in \mathbb{T}_k$, we can use Theorem 1.1 and Lemma 2.4, and get

$$\begin{aligned} I_1 &\leq \sum_{u \in \mathbb{T}_k} \int_{t_k}^\infty \frac{c\nu_u^2}{V_\rho(0, t^{1/2})V_\rho(u, t^{1/2})} dt \\ &\leq |\mathbb{T}_k| \cdot \max_{u \in \mathbb{T}_k} \nu_u^2 \cdot \int_{t_k}^\infty c't^{-(d+\alpha)/(1+\alpha)} dt \\ &\leq 2^{dk} \cdot c''2^{2k\alpha} \cdot c'''(2^{2k(1+\alpha)})^{1-(d+\alpha)/(1+\alpha)} \\ &= c2^{k(2+2\alpha-d)}. \end{aligned}$$

Next, since $(1 \vee \nu_u)\rho(0, u) \geq cs_k$ and $t_k^{1/2} \leq c'|u|^{1+\alpha}$ for $u \in \mathbb{T}_k$, using Theorem 1.1 and Lemma 2.4 again gives

$$\begin{aligned} I_2 &\leq \sum_{u \in \mathbb{T}_k} \int_{s_k}^{t_k} \frac{c\nu_u^2}{V_\rho(0, t^{1/2})V_\rho(u, t^{1/2})} \exp\left(-\frac{\rho(0, u)^2}{ct}\right) dt \\ &\leq \sum_{u \in \mathbb{T}_k} \int_{s_k}^{t_k} \frac{c'\nu_u^2}{t^{(d+\alpha)/(2+2\alpha)} \cdot t^{d/2}|u|^{-(d-1)\alpha}} \exp\left(-\frac{2^{2k(1+\alpha)}}{c't}\right) dt \end{aligned}$$

$$\begin{aligned}
&\leq |\mathbb{T}_k| \cdot \max_{u \in \mathbb{T}_k} \{\nu_u^2 |u|^{(d-1)\alpha}\} \cdot \int_0^\infty c' t^{-(d+\alpha)/(2+2\alpha)-d/2} \exp\left(-\frac{2^{2k(1+\alpha)}}{c' t}\right) dt \\
&\leq 2^{dk} \cdot c'' 2^{(d+1)k\alpha} \cdot (2^{2k(1+\alpha)})^{1-(d+\alpha)/(2+2\alpha)-d/2} \cdot c''' \int_0^\infty x^{(d+\alpha)/(2+2\alpha)+d/2-2} e^{-x} dx \\
&= 2^{k(2+2\alpha-d)} \cdot c'' c''' \int_0^\infty x^{(d+\alpha)/(2+2\alpha)+d/2-2} e^{-x} dx.
\end{aligned}$$

Since $d \geq 3$, we have $\int_0^\infty x^{(d+\alpha)/(2+2\alpha)+d/2-2} e^{-x} dx < \infty$ and so,

$$I_2 \leq c 2^{k(2+2\alpha-d)}.$$

For the remaining term, applying Theorem 1.1 we still have

$$\begin{aligned}
I_3 &\leq \sum_{u \in \mathbb{T}_k} \int_0^{s_k} (\nu_u / \nu_0) \exp\left(-c(\nu_0 \vee \nu_u)^{-1} \rho(0, u) \left(1 \vee \log((\nu_0 \vee \nu_u) \rho(0, u) / t)\right)\right) dt \\
&\leq |\mathbb{T}_k| \cdot s_k \cdot \max_{u \in \mathbb{T}_k} \nu_u \exp\left(-c(\nu_0 \vee \nu_u)^{-1} \rho(0, u)\right) \\
&\leq 2^{dk} \cdot (1 \vee 2^{k\alpha}) 2^{k(1+\alpha)} \cdot 2^{k\alpha} \exp\left(-c'(1 \vee 2^{k\alpha})^{-1} \cdot 2^{k(1+\alpha)}\right) \\
&= 2^{ck} e^{-c' 2^{k(1+\alpha \wedge 0)}} \leq c'' 2^{k(2+2\alpha-d)}.
\end{aligned}$$

Therefore,

$$\mathbf{E}_{0,0}(Z_k) \leq c 2^{k(2+2\alpha-d)}. \quad (6.4)$$

On the other hand, once X and X' collide at some vertex u and some time t , then with at least e^{-2} probability they will stick together during time $[t, t + \nu_u / \mu_u)$, which implies

$$\mathbf{E}_{0,0}(Z_k | Z_k > 0) \geq c \min_{u \in \mathbb{T}_k} \nu_u / \mu_u \geq c' 2^{2k\alpha}.$$

So, for each $k \geq 0$,

$$\mathbb{P}_{0,0}(Z_k > 0) = \frac{\mathbf{E}_{0,0}(Z_k)}{\mathbf{E}_{0,0}(Z_k | Z_k > 0)} \leq c 2^{-(d-2)k}.$$

Therefore,

$$\sum_k \mathbb{P}_{0,0}(Z_k > 0) \leq c \sum_k 2^{-(d-2)k} < \infty.$$

By the Borel-Cantelli Lemma, we completed the proof of (3). \square

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